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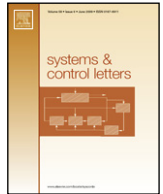
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Dissipativity preserving balancing for nonlinear systems – A Hankel operator approach

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ABSTRACT

In this paper we present a version of balancing for nonlinear systems which is dissipative with respect to a general quadratic supply rate that depends on the input and the output of the system. We discuss an approach that allows us to apply the theory of balancing based upon Hankel singular value analysis. In order to do that we prove that the available storage and the required supply of the original system are the controllability and the observability functions of a modified, asymptotically stable, system. Then Hankel singular value theory can be applied and the axis singular value functions of the modified system equal the nonlinear extensions of “similarity invariants” obtained from the required supply and available storage of the original system. Furthermore, we also consider an extension of normalized coprime factorizations and relate the available storage and required supply with the controllability and observability functions of the factorizations. The obtained relations are used to perform model order reduction based on balanced truncation, yielding dissipative reduced order models for the original systems. A second order electrical circuit example is included to illustrate the results.

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1. Introduction

The problem of model order reduction plays an important role in systems and control theory when dealing with high order and complex models. The idea is to replace such a complex and high order model with a lower order and/or simpler model in order to ease controller design, make controller implementation feasible, or to speed up simulations. Moreover, it is desirable for many properties of the full order system to be preserved in a low order approximation, both from an insight/analysis and a control design point of view. In this paper we are interested in nonlinear model order reduction methods based on balancing while preserving the dissipativity properties of the system. A motivation for preservation of dissipativity stems from the power systems field, [1,2]. The system consists of more machines interconnected through transmission lines yielding complex nonlinear models, which are difficult to analyze and control. A reduced order model that preserves the dissipativity (passivity) of the original system is needed for the analysis of transient stability, and for developing passivity based

controllers. Although easier from a computational point of view, linearization in this case is not an option, since the nonlinearities in the system, such as the coupling between the stator electrical equations and the rotor motion equations or the interconnection with other power systems, play a dominant role.

Basically, dissipativity means that the internal storage of the system never exceeds the storage supplied to the system. It is an important tool for stability analysis in general as described in e.g. [3], or for designing a stabilizing controller with e.g. passivity based control [4]. In this paper, we deal with the class of systems that are dissipative with respect to a quadratic supply rate that depends on the input and the output of the system:

$$s(u, y) = \frac{1}{2} [u^T \quad y^T] J \begin{bmatrix} u \\ y \end{bmatrix}, \quad (1)$$

with $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ and $J \in \mathbb{R}^{(m+p) \times (m+p)}$, such that $J = J^T$. For dissipativity preserving balanced order reduction, we use an extension of the standard balancing concept, introduced for asymptotically stable linear systems in [5] and extended to asymptotically stable nonlinear systems in e.g. [6,7]. In a nutshell a system is called balanced when the required input energy to be supplied to the system in the past to reach a state and the future available energy stored by the same state is balanced and can be (directly) quantified. We aim at an extension of this existing balancing procedure to general dissipative systems, offering a tool to neglect the states of a system that are more dissipative or less dissipative.

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For linear state–space systems, many particular, important cases of a quadratic supply rate (1) have been treated in the literature. For instance, we mention here:

1. the LQG-balancing case and normalized coprime factorization, for unstable systems, i.e. $J = I$ described in [8–11]. Here we include the particular case of H_∞ -balancing, where $J = \begin{bmatrix} I & 0 \\ 0 & (1 - \gamma^{-2})I \end{bmatrix}$, found in e.g. [12,13];
2. the bounded real balancing case, i.e. $J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$, treated in [14–16,11];
3. the positive real balancing case, that is balancing of strictly passive, asymptotically stable, minimal systems, i.e. $J = \begin{bmatrix} I & I \\ 0 & 0 \end{bmatrix}$, treated in [14,17,18].
4. the classic balancing cases can also be taken into account, since the controllability function of an asymptotically reachable system can be considered as a storage function with respect to a supply rate (1) that depends only on the input, i.e. $J = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$. However, the future energy is described by the observed energy at the output after turning off the input (i.e. setting $u = 0$). Details are found in e.g. [5,14,17,19];

In all the above mentioned cases, a pair of positive definite matrices are computed as solutions of particular cases of algebraic Riccati equation. These matrices are then balanced, i.e. brought into equal and diagonal form. The diagonal elements are called singular values and they are similarity invariants. Each singular value is associated to a state component. For model reduction we can choose to truncate the less or the more dissipative states depending on the intended use of the reduced order model.

For nonlinear systems, the classic case was extensively treated in [6,7,20] and developed for the LQG-balancing case, called HJB-balancing, and normalized coprime factorization case in [21,22] as well as for H_∞ case, in [23]. Another balancing perspective is taken in [24], where a time-varying sliding window is used to find a balancing transformation. Here we focus on the former balancing perspective [7,20] valid in neighborhoods (possibly large, and sometimes even global) of equilibrium points. The positive real balancing was extended in [25] to the nonlinear case using the nonlinear balancing method developed in [6] in combination with the passivity theory in e.g. [3,26–29]. In the original case, e.g. [6], the singular values are nonlinear positive functions of the state and in this form they are not invariants, that is they depend on the state–space realization. This drawback is solved for the classic asymptotically stable balancing case, in [7,20]. Here, the Hankel operator gain structure is investigated and a set of invariant, with respect to a coordinate transformation, (axis) singular value functions for the system are derived. For each component the axis singular value function measures the input effort to steer to that state component from the past and the amount of energy that it provides in the future at the output. Thus the input–output point of view from the linear systems is extended to the nonlinear systems case rendering a more precise balancing procedure.

The aim of this paper is to give a general version of the theory of balanced truncation for nonlinear, dissipative systems, that also includes the nonlinear versions of some of the particular cases listed above from 1 to 3. In detail, the problem is stated as follows: given a nonlinear system, dissipative with respect to (1) find a coordinate transformation such that in the new coordinates, the states are ordered according to the amount of storage that they dissipate. Then, by truncating the more/less dissipative states a reduced order model is obtained, which is in its turn dissipative with respect to the same supply rate. This paper emphasizes the theoretical results of such an approach rather than discuss its numerical aspects. However, we will briefly mention some of the computational aspects encountered while applying such a technique, and illustrate these issues with an example.

A similar problem has been previously addressed in the work of Weiland [13] and more recently in the work of Minh [30], for linear systems. Their problem and the general results therein are described in the linear behavioural framework.

We approach the nonlinear problem by combining the balancing technique developed in [7,20] with the dissipativity theory described in e.g. [3,26]. The latter provides the available storage and the required supply functions as the solutions of a Hamilton–Jacobi–Bellman equation. Unfortunately, an operator interpretation of these energy functions, such as we have for the Hankel operator, is missing. Still, we prove that they are in fact the observability and the controllability functions, respectively, of an extended system. For this system, the Hankel gain structure problem can be solved and the solution provides the (coordinate-free) axis singular value functions of the original system. Finally, balanced realizations of the original system are obtained.

Based on these results, cases like passive (positive real) or bounded real balanced truncation are extended to the nonlinear case, resulting in nonlinear reduced models that preserve passivity or bounded realness, respectively.

In Section 2, a brief overview of the dissipativity theory is given with respect to a nonlinear system, as well as of the storage functions, the available storage and the required supply. Section 3 deals with constructing the new state–space realization whose controllability and observability function are the storage functions defined for the original dissipative system. Also, the particular cases of passivity and bounded realness are treated. Section 3.2 deals with a factorization approach where a new pair of storage functions is derived. In Section 4, the relation between the axis singular value functions of the original system, and the axis singular value functions defined with respect to the available storage and the required supply is provided. Also the balanced realizations are written and model reduction is performed. We will illustrate the results with an example in Section 5. Some conclusions and future work make up Section 6.

Notation. Let $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^m$ be two signals defined as:

$$u(t) = [u_1(t) \ u_2(t) \ \dots \ u_m(t)]^T$$

and

$$y(t) = [y_1(t) \ y_2(t) \ \dots \ y_m(t)]^T.$$

The norm of $u(t)$ is given by

$$\|u(t)\| = \sqrt{u^T(t)Mu(t)}.$$

We denote by $\|u(t)\|_M^2 = u^T(t)Mu(t)$, $M \in \mathbb{R}^{m \times m}$. Let $L_2^m([a, b])$ be the set of signals with positive support and finite L_2 -norm, defined as $\|u(t)\|_{L_2} = \sqrt{\int_a^b \|u(t)\|^2 dt}$. Let $x \in \mathbb{R}^n$ and $F : \mathbb{R}^n \rightarrow \mathbb{R}$ be a differentiable function, then we write

$$\frac{\partial F}{\partial x}(a) = \left[\frac{\partial F}{\partial x_1}(a) \ \frac{\partial F}{\partial x_2}(a) \ \dots \ \frac{\partial F}{\partial x_n}(a) \right]$$

and

$$\frac{\partial^T F}{\partial x}(a) = \left[\frac{\partial F}{\partial x_1}(a) \ \frac{\partial F}{\partial x_2}(a) \ \dots \ \frac{\partial F}{\partial x_n}(a) \right]^T, \quad a \in \mathbb{R}^n.$$

Throughout the rest of the paper we will make use of the operator $d(\cdot)$, called Fréchet derivative, defined as follows: given a function $f : X \rightarrow Y$, with X and Y Banach spaces, the Fréchet derivative of f is df which satisfies

$$f(x + \epsilon) - f(x) = df(x)(\epsilon) + o(\|\epsilon\|)$$

and $df(x)(\epsilon)$ is linear in ϵ .

2. Preliminaries

We treat the following class of nonlinear systems:

$$\begin{aligned}\dot{x} &= f(x) + g(x)u \\ y &= h(x) + d(x)u,\end{aligned}\quad (2)$$

where $x \in \mathbb{R}^n$, is the state vector, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, are the input vector and the output vector of the dynamic system, respectively. We assume $f(x)$, $g(x)$, $h(x)$ and $d(x)$ are smooth.

2.1. Dissipativity and storage functions

The property of a system being dissipative with respect to the supply rate (1) is described by the following:

Definition 1 ([3,26]). A system (2) is called dissipative with respect to a supply rate $s(u, y)$, if there exists a function $S : \mathbb{R}^n \rightarrow \mathbb{R}$, $S(x) \geq 0$, such that:

$$S(x_0) + \int_{t_0}^{t_1} s(u, y) dt \geq S(x_1), \quad (3)$$

for all x, u and $t_1 \geq t_0$, with $x_0 = x(t_0)$ and $x_1 = x(t_1)$, the state of the system (2) at t_1 , resulting from the initial condition x_0 and input u . Any function $S(x)$ that satisfies inequality (3) is called a storage function. \square

Inequality (3) is called the dissipation inequality. It expresses that the storage at time t_1 is at most equal to the sum of the storage at time t_0 and the external supply between t_0 and t_1 . Hence, the system can only dissipate storage.

Remark 1. Assuming S is at least continuously differentiable, inequality (3) can be also written in differential form as (see e.g. [26]):

$$\frac{\partial S(x)}{\partial x} (f(x) + g(x)u) \leq s(u, y), \quad (4)$$

called the differential dissipation inequality. \square

We are interested in the study of two particular energy storage functions: the available stored energy at the state x_0 and the required supply at the state x_0 .

Definition 2 ([3,14]). The available storage function of a system (2) is the function:

$$S_a(x_0) = \sup_{u \in L_2(0, \infty)} - \int_0^\infty s(u(t), y(t)) dt, \quad (5)$$

$$x(0) = x_0, \quad x(\infty) = 0.$$

The required supply function of system (2) is the function:

$$S_r(x_0) = \inf_{u \in L_2(-\infty, 0)} \int_{-\infty}^0 s(u(t), y(t)) dt, \quad (6)$$

$$x(0) = x_0, \quad x(-\infty) = 0,$$

subject to the constraints (2). \square

$S_a(x)$ represents the maximal amount of storage that can be extracted from the system when starting at the initial state x_0 . The minus sign indicate that we are concerned with the storage coming out of the system. $S_r(x)$ represents the minimal amount of storage required to be supplied to the system in order to reach x_0 from the equilibrium. These functions are both storage functions in the sense of Definition 1, see [26]. We make the following working assumption:

Assumption 1. 0 is an equilibrium point of the system and furthermore, $h(0) = 0$.

The property of the system being reachable from the equilibrium 0 is a condition for the existence and nonnegativity of the storage functions defined in relations (5) and (6), see e.g. [31,32,3]. So, we introduce the following:

Assumption 2. The system (2) is asymptotically reachable from 0.³

Lemma 1 ([31,26,33]). Let system (2) be dissipative with respect to $s(u, y)$ as in Definition 1 and satisfy Assumption 2. Then, the energy functions S_a and S_r as in Definition 2 exist and fulfill $0 \leq S_a \leq S_r$. If the inequality (3) is strict, then $S_a < S_r$. \square

For stating the positive definiteness of the storage functions of a dissipative system, additional assumptions are required, see e.g., [33].

Assumption 3. There exists $\varphi(\cdot)$, such that $s(\varphi(y), y) < 0$, $(\forall) y$, $\varphi(0) = 0$.

This basically means that we assume we can find an input ensuring that the storage flows out of the system.

Assumption 4. The system is zero-state observable.⁴

Lemma 2 ([33,27]). Assume system (2) is dissipative with respect to supply rate (1) and Assumptions 3 and 4 hold. Then any storage function S that fulfills (3) is strictly positive definite. \square

Special cases of storage functions. The cases of LQG/HJB-balancing and coprime factorization balancing, from e.g. [21], do not satisfy Assumption 3, due to the definition of their supply rate, i.e. $s(u, y) = \|u\|^2 + \|y\|^2$ ($J = I$). In this case, the system is dissipative with respect to the supply rate $s(u, y)$ with the required supply defined as $S_r(x) = -K^-(x_0)$ and the available storage $S_a(x_0) = -K^+(x_0) \leq 0$, where K^+ and K^- are described by the following relations:

$$\begin{aligned}K^-(x_0) &= \min_{u \in L_2(-\infty, 0)} \frac{1}{2} \int_{-\infty}^0 \|u(t)\|^2 + \|y(t)\|^2 dt, \\ x(-\infty) &= 0, \quad x(0) = x_0 \\ K^+(x_0) &= \min_{u \in L_2(0, \infty)} \frac{1}{2} \int_0^\infty \|u(t)\|^2 + \|y(t)\|^2 dt, \\ x(\infty) &= 0, \quad x(0) = x_0.\end{aligned}\quad (7)$$

$K^-(x_0)$ is called the past input energy and $K^+(x_0)$ is the future energy of the system at state x_0 . Another case that is slightly different from the general approach is the H_∞ case, as in e.g. [23], where $J = \begin{bmatrix} I & 0 \\ 0 & (1 - \gamma^{-2})I \end{bmatrix}$. Here the required supply and the available storage are $S_r(x_0) = Q_\gamma^-(x_0)$ and $S_a(x_0) = (1 - \gamma^{-2}) Q_\gamma^+(x_0)$, for $\gamma > 1$ and $S_a(x_0) = (\gamma^{-2} - 1) Q_\gamma^+(x_0)$, for $\gamma < 1$, where Q_γ^- and Q_γ^+ are defined as:

$$\begin{aligned}Q_\gamma^-(x_0) &= \min_{u \in L_2(-\infty, 0)} \frac{1}{2} \int_{-\infty}^0 (1 - \gamma^{-2}) \|y(t)\|^2 + \|u(t)\|^2 dt, \\ x(-\infty) &= 0, \quad x(0) = x_0, \quad \forall \gamma \\ Q_\gamma^+(x_0) &= \min_{u \in L_2(0, \infty)} \frac{1}{2} \int_0^\infty \|y(t)\|^2 + \frac{1}{1 - \gamma^{-2}} \|u(t)\|^2 dt, \\ x(\infty) &= 0, \quad x(0) = x_0, \quad \text{for } \gamma > 1\end{aligned}$$

³ The system (2) is called asymptotically reachable from 0 if for all \bar{x} there exists an input $u \in L_2(0, \infty)$ and $t \geq 0$ such that $\bar{x} = \lim_{t \rightarrow \infty} \phi(t, 0, 0, u)$.

⁴ The system (2) is zero-state observable if $u(t) \equiv 0$, $y(t) \equiv 0$ implies $x(t) \equiv 0$.

while if $\gamma < 1$,

$$Q_{\gamma}^{+}(x_0) = \max_{u \in L_2(0, \infty)} \frac{1}{2} \int_0^{\infty} \|y(t)\|^2 dt + \frac{1}{1 - \gamma^{-2}} \|u(t)\|^2 dt, \\ x(\infty) = 0, \quad x(0) = x_0.$$

In this case, the functions that are later used for balancing are Q_{γ}^{-} and Q_{γ}^{+} , i.e. the required supply and a modified available storage, determined by the value of γ .

2.2. The Hankel singular value problem and the axis singular value functions

In this section we briefly describe the Hankel singular value problem for a nonlinear system, as shown in the results from [7,20], useful for obtaining a balanced realization. We consider system (2) such that [Assumption 1](#) holds. Assuming that the equilibrium 0 is asymptotically stable, we can define a couple of operators and energy functions. First, the controllability operator is defined as $\mathcal{C} : L_2^m(\mathbb{R}_+) \rightarrow \mathbb{R}^n$, $\mathcal{C}(u) = x_0$. The observability operator is $\mathcal{O} : \mathbb{R}^n \rightarrow L_2^m(\mathbb{R}_+)$, $\mathcal{O}(x_0) = y$. Then the Hankel operator, mapping past inputs to future outputs, is defined by the relation

$$\mathcal{H} = \mathcal{O}\mathcal{C}.$$

Accordingly, the controllability and observability energy functions of (2) are described by

$$L_c(x_0) = \min_{u \in L_2(-\infty, 0), x(0)=x_0, x(-\infty)=0} \int_{-\infty}^0 \frac{1}{2} \|u(t)\|^2 dt \\ = \frac{1}{2} \|\mathcal{C}^{\dagger}(x_0)\|_{L_2}^2 \quad (8)$$

$$L_o(x_0) = \left\{ \int_0^{\infty} \frac{1}{2} \|y(t)\|^2 dt, x(0) = x_0, u = 0 \right\} \\ = \frac{1}{2} \|\mathcal{O}(x_0)\|_{L_2}^2. \quad (9)$$

By \mathcal{C}^{\dagger} we denote the pseudo-inverse of the controllability operator, namely $\mathcal{C}^{\dagger} : L_2^m(\mathbb{R}_+) \rightarrow \mathbb{R}^n$, $\arg \inf_{\mathcal{C}(u)=x_0} u = \mathcal{C}^{\dagger}(x_0)$. Under [Assumption 1](#) and appropriate existence conditions, the controllability function is the unique antistabilizing solution of a Hamilton–Jacobi–Bellman equation. Similarly, the observability function is the unique solution of a nonlinear Lyapunov equation. Furthermore, if the system satisfies [Assumption 2](#), then the controllability function is positive and, if [Assumption 4](#) holds, then the observability function is positive, respectively.

Theorem 1 ([6]). Consider system (2) asymptotically stable about 0. Assume L_c exists and is smooth. Then L_c is the unique solution of the Hamilton–Jacobi–Bellman equation:

$$\frac{\partial L_c(x)}{\partial x} f(x) + \frac{1}{2} \frac{\partial L_c(x)}{\partial x} g(x) g^T(x) \frac{\partial L_c(x)}{\partial x} = 0, \quad L_c(0) = 0 \quad (10)$$

such that $-(f(x) + g(x)g^T(x) \frac{\partial L_c(x)}{\partial x})$ is asymptotically stable. If the system satisfies [Assumption 2](#), then $L_c(x_0) > 0$, $x_0 \neq 0$. Assume L_o exists and is smooth. Then L_o is the unique solution of the nonlinear Lyapunov equation:

$$\frac{\partial L_o(x)}{\partial x} f(x) + \frac{1}{2} h^T(x) h(x) = 0, \quad L_o(0) = 0. \quad (11)$$

If the system satisfies [Assumption 4](#), then $L_o(x_0) > 0$, $x \neq 0$. \square

The starting point in solving the Hankel singular value problem is the investigation of the gain structure of the Hankel operator. The gain structure problem means examining the largest singular value, where a singular value is defined as

$$\sigma(u) = \frac{\|\mathcal{H}(u)\|_{L_2}}{\|u\|_{L_2}}. \quad (12)$$

According to [20], the following problem must be solved:

$$d \frac{\|\mathcal{H}(u)\|_{L_2}}{\|u\|_{L_2}} du = 0, \quad \text{subject to } u \in \text{Im } \mathcal{C}^{\dagger}, \quad (13)$$

that characterizes all the critical points u as well as the optimal one that gives the largest eigenvalue. This problem has the alternative formulation: there exists $\lambda \in \mathbb{R}$ s.t.

$$(d\mathcal{H}(u))^* \mathcal{H}(u) = \lambda u, \quad (14)$$

where $(d\mathcal{H}(u))^*$ represents the adjoint of the linear operator $d\mathcal{H}(u)$, see [7]. The problem of finding $u \in \text{Im } \mathcal{C}^{\dagger}$ such that Eq. (14) is satisfied, is called the Hankel singular value problem. A characterization, in terms of the controllability and observability function derivatives, of the Eq. (14) is described in the following result.

Theorem 2 ([20]). Assume that the controllability operator, its pseudo-inverse and the observability operator exist and are continuously differentiable. Assume moreover, that there exist $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^n$ satisfying

$$dL_o(x) = \lambda \cdot dL_c(x). \quad (15)$$

Then if

$$u \in \text{Im } \mathcal{C}^{\dagger}, \quad (16)$$

λ satisfies the Hankel singular value Eq. (14). \square

Remark 2. In the linear case, for a minimal asymptotically stable system, if $W > 0$ and $M > 0$ denote the controllability and observability Gramians, respectively, then $L_c(x) = \frac{1}{2} x^T W^{-1} x$ and $L_o(x) = \frac{1}{2} x^T M x$. Thus, Eq. (15) is equivalent to $x^T M = \lambda x^T W^{-1}$, further equivalent to $WMx = \lambda x$, i.e. the squared Hankel singular values are the eigenvalues of WM . \square

There exists a solution for the nonlinear Hankel singular value problem, depicted in the following theorem:

Theorem 3 ([7]). Suppose that the linearization of (2) has nonzero distinct Hankel singular values. Then, there exists a neighbourhood $U \subset \mathbb{R}$ of 0 and n smooth functions $\rho_i : U \rightarrow (0, \infty)$, $\rho_i(s) > 0$, $i = 1, \dots, n$ such that: $\min\{\rho_i(s), \rho_i(-s)\} \geq \max\{\rho_{i+1}(s), \rho_{i+1}(-s)\}$ holds for all $s \in U$, $i = 1, \dots, n-1$. Moreover, there exist $\xi_i(s)$, $i = 1, \dots, n$, $\xi_i : S_0 \rightarrow X$, $S_0 \subset \mathbb{R}$ satisfying the following:

$$L_c(\xi_i(s)) = \frac{s^2}{2}, \quad L_o(\xi_i(s)) = \frac{\rho_i^2(s)s^2}{2}, \\ \frac{\partial L_o}{\partial x}(\xi_i(s)) = \lambda_i(s) \frac{\partial L_c}{\partial x}(\xi_i(s)), \quad (17) \\ \lambda_i(s) = \rho_i^2(s) + \frac{s}{2} \frac{d\rho_i^2(s)}{ds}.$$

Furthermore $\rho_i(s) = \sigma(u)|_{u=\mathcal{C}^{\dagger}(\xi_i(s))}$, with σ defined by relation (12). Moreover, if $S_0 = \mathbb{R}$, the Hankel norm of the system is $\sup_s \rho_1(s)$. \square

A new input-normal output-diagonal realization can be obtained, where all the coordinate axes of the state-space appear separately in the observability and controllability functions.

Theorem 4 ([20]). Let system (2) be such that its linearization is minimal and satisfies the condition in the preamble of Theorem 3. Then there exists a coordinate transformation $x = \Phi(z)$ such that:

$$\begin{aligned} L_c(\Phi(z)) &= \frac{1}{2} z^T z, \\ L_o(\Phi(z)) &= \frac{1}{2} (z_1^2 \rho_1^2(z_1) + z_2^2 \rho_2^2(z_2) + \dots + z_n^2 \rho_n^2(z_n)). \end{aligned} \quad (18)$$

and $\Phi(0, \dots, z_i, \dots, 0) = \xi_i(z_i)$. \square

Finally, a balanced realization can be written, based on the Hankel singular value problem:

Theorem 5 ([20]). Let system (2) be such that its linearization is minimal and satisfies the condition in the preamble of Theorem 3. Then there exists a neighbourhood X of the origin and a coordinate transformation $x = \Phi(z)$ such that the energy functions become:

$$L_c(\Phi(z)) = \frac{1}{2} \sum_{i=1}^n \frac{z_i^2}{\rho_i(z_i)} \quad \text{and} \quad L_o(\Phi(z)) = \frac{1}{2} \sum_{i=1}^n z_i^2 \rho_i(z_i). \quad (19)$$

If $U = \mathbb{R}^n$ then the Hankel norm of the system is given by $\sup_{z_1} \rho_1(z_1)$. \square

Next, we adapt these results to the general case of dissipative systems. Namely, we give a Hankel operator interpretation to the storage functions and to the dissipation analysis we perform on the states. We turn the problem of how much storage a state dissipates into the analysis of the gain between input and output energy of the same state as part of the dynamics of an extended (diluted) system. Hence, the required storage to be supplied to the original system in order to reach the state is the same as the minimum amount of control energy required by the extended system to reach the same state. Similarly, the available storage at a state will represent the (weighted) output energy of the extended system provided by the same state. Thus, the Hankel singular value problem of the extended system is related to the balancing problem with respect to the available storage and required supply storage functions corresponding to the original system.

3. Storage functions of state-space realizations

In this section we define the to-be-balanced storage functions and place them in the context of a controllability and observability analysis of extended systems.

We consider a nonlinear system (2) that is dissipative with respect to the supply rate (1). We present the available storage and the required supply defined by Eqs. (5) and (6) as the solutions of a Hamilton–Jacobi–Bellman equation.

Denote by

$$c(x) = [I \quad d^T(x)] \begin{bmatrix} 0 \\ h(x) \end{bmatrix} \quad (20)$$

and by

$$r(x) = [I \quad d^T(x)] \begin{bmatrix} I \\ d(x) \end{bmatrix} \quad (21)$$

of dimension $m \times m$. We make an additional assumption:

Assumption 5. Matrix $r(x)$ is positive definite.

The storage functions are computed as the stabilizing and anti-stabilizing solution, respectively, of a (general) Hamilton–Jacobi–Bellman equation, which is the nonlinear generalization of the general Algebraic Riccati equation as in [15].

Theorem 6. Let system (2) satisfy Assumption 2 and let $r(x)$ satisfy Assumption 5. Then the Hamilton–Jacobi–Bellman equation:

$$\begin{aligned} \frac{\partial S(x)}{\partial x} f(x) + \frac{1}{2} \left(\frac{\partial S(x)}{\partial x} g(x) - c^T(x) \right) r^{-1}(x) \\ \times \left(g^T(x) \frac{\partial S(x)}{\partial x} - c(x) \right) - \frac{1}{2} [0 \quad h^T(x)] \begin{bmatrix} 0 \\ h(x) \end{bmatrix} = 0 \end{aligned} \quad (22)$$

has the smooth solution $S_a(x)$, $S_a(0) = 0$, such that

$$f(x) + g(x) r^{-1}(x) \left(g^T(x) \frac{\partial S_a(x)}{\partial x} - c(x) \right) \quad (23)$$

is asymptotically stable and the smooth solution $S_r(x)$, $S_r(0) = 0$, such that

$$- \left(f(x) + g(x) r^{-1}(x) \left(g^T(x) \frac{\partial S_r(x)}{\partial x} - c(x) \right) \right) \quad (24)$$

is asymptotically stable. \square

Proof. The detailed proof is similar to [25, Theorem 11] and can be found in [34, Chapter 3, Section 3.1.]. Here we present a sketch of the proof. Since system (2) is dissipative with respect to the supply rate $s(u, y)$ and reachable, according to Lemma 1, $S_a(x(t))$ and $S_r(x(t))$ exist and are nonnegative. We develop the proof for $S_r(x)$. The sequel follows the idea in Scherpen [6, Theorem 3.2]. By definition, $S_r(x) = \inf_{u, x(-\infty)=0} \int_{-\infty}^t s(u(s), y(s)) ds$. Because $S_r(x)$ exists, there exists an optimal input u^* , i.e. $S_r(x(t)) = \int_{-\infty}^t s(u^*(s), y^*(s)) ds$, where $y^*(s)$ is the output of the system with the input u^* . Differentiating $S_r(x(t))$ with respect to time we get:

$$\begin{aligned} \dot{S}_r(x(t)) &= s(u^*, y^*) \Rightarrow \frac{\partial S_r(x)}{\partial x} \\ &\times (f(x) + g(x) u^*) - s(u^*, y^*) = 0. \end{aligned} \quad (25)$$

Furthermore, one can compute that:

$$\begin{aligned} \dot{S}_r(x) &= s(u, y) \\ &- \frac{1}{2} \left\| u - r^{-1}(x) \left(g^T(x) \frac{\partial S_r(x)}{\partial x} - c(x) \right) \right\|_{r(x)}^2. \end{aligned} \quad (26)$$

Relation (25), can be written as $\frac{\partial S_r(x)}{\partial x} f(x) + \frac{\partial S_r(x)}{\partial x} g(x) u^* - s(u^*, y^*) = 0$. Relations (26) and (22) give:

$$\begin{aligned} \frac{1}{2} s(u^*, y^*) - \frac{\partial S_r(x)}{\partial x} g(x) u^* \\ = -\frac{1}{2} \left(\frac{\partial S_r(x)}{\partial x} g(x) - c^T(x) \right) r^{-1}(x) \left(g^T(x) \frac{\partial S_r(x)}{\partial x} - c(x) \right) \\ + \frac{1}{2} \left\| u - r^{-1}(x) \left(g^T(x) \frac{\partial S_r(x)}{\partial x} - c(x) \right) \right\|_{r(x)}^2 + \frac{1}{2} q(x). \end{aligned}$$

Then we compute

$$\begin{aligned} \frac{\partial S_r(x)}{\partial x} f(x) + \frac{1}{2} \left(\frac{\partial S_r(x)}{\partial x} g(x) - c^T(x) \right) r^{-1}(x) \\ \times \left(g^T(x) \frac{\partial S_r(x)}{\partial x} - c(x) \right) + \frac{1}{2} q(x) \\ - \frac{1}{2} \left\| u^* - r^{-1}(x) \left(g^T(x) \frac{\partial S_r(x)}{\partial x} - c(x) \right) \right\|_{r(x)}^2 = 0. \end{aligned} \quad (27)$$

It follows that $u^* = r^{-1}(x) \left(g^T(x) \frac{\partial S_r(x)}{\partial x} - c(x) \right)$ is the optimal control we are looking for. Similar steps can be taken to proof the result for S_a . \square

Remark 3. In the linear case, consider (A, B, C, D) a minimal realization. Then $S_a(x) = \frac{1}{2}x^T K_{\min} x$ and $S_r = \frac{1}{2}x^T K_{\max} x$, with K_{\min} and K_{\max} the stabilizing and antistabilizing solutions of the general algebraic Riccati equation (see [15]):

$$A^T K + KA + \left(KB - [0 \quad C^T] \begin{bmatrix} I \\ D \end{bmatrix} \right) \left([I \quad D^T] \begin{bmatrix} I \\ D \end{bmatrix} \right)^{-1} \times \left(B^T K - [I \quad D^T] \begin{bmatrix} 0 \\ C \end{bmatrix} \right) - [0 \quad C^T] \begin{bmatrix} 0 \\ C \end{bmatrix} = 0.$$

Substituting different values for J , as described in Section 1, one recognizes the positive real Riccati equation, the bounded real Riccati equation, etc. \square

Remark 4. If we define the following Hamiltonian function:

$$H(x, p, u) = p^T [f(x) + g(x)u] - \frac{1}{2} [u^T \quad y^T] \begin{bmatrix} u \\ y \end{bmatrix}, \quad (28)$$

then, according to [26, Chapter 7], the optimal control that solves the problem in (5) and (6) satisfies the condition $\frac{\partial H(x, p, u)}{\partial u} = 0$.

Also, note that the condition $\frac{\partial^2 H(x, p, u)}{\partial u^2} = r(x) > 0$, as in Assumption 5 (see e.g. [26]) renders the optimal control problems in (5) and (6), non-singular. If $r(x)$ is singular, the available storage and the required supply still exists, but they are no longer the solutions of the Hamilton–Jacobi–Bellman equation (22). They can be computed, for instance, by discretizing the state–space system and solving a corresponding nonlinear optimization problem (based on the definitions of the storage functions), for more details see [35]. \square

We recall an important property for solutions of the Hamilton–Jacobi–Bellman equation (22), found in e.g. [26,3]. It essentially says that when there is dissipation, then the available stored energy is less than the energy supply required.

Theorem 7 ([26, Prop. 7.1.8, Rem. 7.1.9]). Assume that matrix Z defined by

$$Z = \begin{bmatrix} \frac{\partial^2 H}{\partial x \partial p}(0, 0) & \frac{\partial^2 H}{\partial p^2}(0, 0) \\ -\frac{\partial^2 H}{\partial x^2}(0, 0) & -\frac{\partial^2 H}{\partial p \partial x}(0, 0) \end{bmatrix}, \quad (29)$$

H as in (28), has no eigenvalues on the imaginary axis. Then, any function $S(x)$ that satisfies (22) fulfills $S_a \leq S(x) \leq S_r(x)$. Moreover, $S_a(x) < S_r(x)$. \square

3.1. A direct controllability and observability approach

We present a direct connection of the required supply and the available storage functions to the controllability and observability functions of an extended system, respectively. However, this approach is bound by assumptions, as mentioned in Section 2.1, where special cases of storage functions are discussed. Here, we describe two of the most important cases from literature, namely the positive real (passivity) and the bounded real case.

In order to make the extended system, we first rewrite Eq. (22) as:

$$\frac{\partial S}{\partial x} (f(x) - g(x)r^{-1}(x)c(x)) + \frac{1}{2} \frac{\partial S}{\partial x} g(x)r^{-1}(x)g^T(x) \frac{\partial^T S}{\partial x} + \frac{1}{2} c^T(x)r^{-1}(x)c(x) - \frac{1}{2} [0 \quad h^T(x)] \begin{bmatrix} 0 \\ h(x) \end{bmatrix} = 0. \quad (30)$$

Doing a regrouping of terms and using (20), we get:

$$\frac{\partial S}{\partial x} (f(x) - g(x)r^{-1}(x)c(x)) + \frac{1}{2} \frac{\partial S}{\partial x} g(x)r^{-1}(x)g^T(x) \frac{\partial^T S}{\partial x} + \frac{1}{2} [0 \quad h^T(x)] L(x) \begin{bmatrix} 0 \\ h(x) \end{bmatrix} = 0, \quad (31)$$

where $L(x) = J \begin{bmatrix} I \\ d(x) \end{bmatrix} r^{-1}(x) [I \quad d^T(x)] J - J$. The sign of $L(x)$ cannot be determined in general, but we study cases where $L(x) > 0$, in particular the positive and bounded real cases.

3.1.1. The passivity case

In this section, we treat the particular case of passivity since model order reduction preserving this property is useful in applications such as power systems stability/passivity analysis and controller design, as well as the application of electrical circuit simulators. In this case, J from Eq. (1), and $r(x)$ and $c(x)$ from Eqs. (21) and (20), respectively, are given by:

$$J = \begin{bmatrix} 0 & I_{m \times m} \\ I_{m \times m} & 0 \end{bmatrix}, \quad r(x) = d(x) + d^T(x) \quad \text{and} \quad c(x) = h(x).$$

We assume that the number of inputs m equals the number of outputs p . The supply rate is $s(u, y) = u^T y$ and $L(x) > 0$. Since for many physical systems the scalar product $u^T y$ represents the supplied power, its integral represents the internal energy. Therefore, we call storage functions that satisfy (3) with supply rate $u^T y$, energy functions.

The notion of passivity is related to the notion of positive realness, i.e. the energy is always positive. We briefly present these ideas:

Definition 3 ([28]). A system (2) is called positive real if, for all $u \in L_2^p(\mathbb{R})$ and $t \geq 0$,

$$\int_0^t u(\tau)^T y(\tau) d\tau \geq 0, \quad (32)$$

whenever $x(0) = 0$. \square

Combined with Lemma 1, we obtain the link between passivity and positive realness:

Proposition 1 ([28]). A passive system (2) is positive real. Conversely, a positive real system (2), that satisfies Assumption 2, is passive. \square

In the sequel, we treat the case of a system being positive real and reachable, so equivalently passive, with $r(x) > 0$. This case is actually similar to the strictly input passive case, see van der Schaft [26] for more details.

Linear systems case. Consider a linear system: $\dot{x} = Ax + Bu$, $y = Cx + Du$, where A, B, C, D are constant matrices of appropriate dimensions, and $R = D + D^T$. We give a brief overview of the positive real balanced truncation technique that yield positive real reduced order models.

Definition 4 ([17,14]). A linear, square, asymptotically stable system $G(s) = C(sI - A)^{-1}B + D$ is called positive real if

$$G^T(-j\omega) + G(j\omega) \geq 0, \quad \forall \omega \in \mathbb{R}.$$

If the inequalities are strict then the system is called strictly positive real. \square

Strict positive realness can be studied with the Kalman–Yakubovich–Popov lemma, see e.g. [14]. The energy functions are quadratic and related to a pair of matrices called the positive real Gramians of the system.

Theorem 8 ([3]). Assume (A, B, C, D) is reachable and observable (minimal) and strictly positive real. Then $S_a(x) = \frac{1}{2}x^T K_{\min} x$ and $S_r(x) = \frac{1}{2}x^T K_{\max} x$, where $K_{\min} > 0$ and $K_{\max} > 0$ are the minimal stabilizing, respectively maximal antistabilizing solutions of the Positive Real Algebraic Riccati equation:

$$KA + A^T K + (KB - C^T)R^{-1}(B^T K - C) = 0. \quad \square \quad (33)$$

Definition 5 ([14]). A strictly positive real linear system is called positive real balanced if $K_{\min} = K_{\max}^{-1} = \text{diag}(\pi_1, \pi_2, \dots, \pi_n)$, where $1 \geq \pi_1 > \pi_2 > \dots > \pi_n > 0$ are the positive real singular values. \square

Let (A, B, C, D) be a positive real balanced system. Assume that $\pi_k > \pi_{k+1}$ and accordingly partition the state-space realization into:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C = [C_1 \ C_2],$$

$$A_{11} \in \mathbb{R}^{k \times k}, \quad B_1 \in \mathbb{R}^{k \times m}, \quad C_1 \in \mathbb{R}^{p \times k}.$$

Theorem 9 ([14,17]). Let the reduced order model

$$G_r(s) = C_1(sI - A_{11})^{-1}B_1 + D$$

be obtained by truncation. Then (A_{11}, B_1, C_1) is asymptotically stable, minimal and strictly positive real. \square

Nonlinear positive real/passivity case. In the nonlinear case, the above linear positive real balanced truncation is less clear, since there is no definition yet of the axis positive real singular value functions, similar to the axis singular value functions based on the Hankel operator, like in [7]. Therefore, we now reformulate the positive real balancing problem of system (2) into a Hankel balancing problem of an extended system. Eq. (22) takes the form presented in [25, Theorem 11].

$$\frac{\partial S}{\partial x} f(x) + \frac{1}{2} \left(\frac{\partial S}{\partial x} g(x) - h^T(x) \right) r^{-1}(x) \times \left(\frac{\partial S}{\partial x} g(x) - h^T(x) \right)^T = 0 \quad (34)$$

or, equivalently, rewriting Eq. (31):

$$\frac{\partial S}{\partial x} (f(x) - g(x)r^{-1}(x)h(x)) + \frac{1}{2} \frac{\partial S}{\partial x} g(x)r^{-1}(x)g^T(x) \frac{\partial^T S}{\partial x} + \frac{1}{2} h^T(x)r^{-1}(x)h(x) = 0. \quad (35)$$

The energy functions, the available storage and the required supply of system (2), can be written as the observability and controllability functions of another system, as follows:

Theorem 10. Assume that system (2) satisfies Assumptions 2, 4. Moreover, assume that the system (2) is passive and Assumptions 3, 5 hold. Additionally, assume that $f(x) - g(x)r^{-1}(x)h(x)$ is asymptotically stable, then $S_a(x)$ is the observability function of the following system:

$$\begin{aligned} \dot{x} &= f(x) - g(x)r^{-1}(x)h(x) \\ y_1 &= -r^{-1}(x)g^T(x) \frac{\partial S_a(x)}{\partial x} \\ y_2 &= r^{-1}(x)h(x). \quad \square \end{aligned} \quad (36)$$

Proof. Since the system (2) is assumed passive, reachable and zero-state observable (satisfying Assumption 3), then $S_a(x) > 0$ exists and satisfies Eq. (34) and equivalently Eq. (35). Also, if we assume that $y_2 = 0$ then, due to $r(x) > 0$, we have $h(x) = 0$. Since (2) is assumed zero-state observable, we get $x(t) = 0$, meaning that (36) is zero-state observable, too. Since $f - gr^{-1}h$ is assumed asymptotically stable, and passivity ensures that Eq. (35) rewritten as

$$\frac{\partial \bar{L}_o(x)}{\partial x} (f(x) - g(x)r^{-1}(x)h(x)) + \frac{1}{2} [y_1^T \ y_2^T] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 0$$

has a unique stabilizing solution S_a , it immediately follows that S_a equals the observability function

$$\bar{L}_o(x) = \int_0^\infty \frac{1}{2} \left\| \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \right\|^2 dt, \quad x(0) = x, \\ x(\infty) = 0, \quad \bar{L}_o(x) > 0$$

of system (36). Thus, $S_a(x) = \bar{L}_o(x)$, for all $x \in W$, which proves the statement of the theorem. \square

Remark 5. From [6] it follows that asymptotic stability of (36) is ensured by strict positivity of S_a and the zero-state observability of system (36). \square

Following the same line of thinking, we can prove that the required supply of the passive system (2) is the controllability function of an extended system.

Theorem 11. Assume system (2) is asymptotically reachable from 0, zero-state observable and passive and Assumption 5 holds. Then $S_r(x)$ is the controllability function of the following system:

$$\dot{x} = f(x) - g(x)r^{-1}(x)h(x) + g(x)r^{-\frac{1}{2}}(x)u_1 + K(x)r^{-\frac{1}{2}}(x)u_2, \quad (37)$$

where $K(x)$ satisfies $\frac{\partial S_r(x)}{\partial x} K(x) = h^T(x)$. \square

Proof. Since the system (2) is assumed passive, asymptotically reachable from 0 and zero-state observable then $S_r(x) > 0$ exists and satisfies Eq. (34) and equivalently Eq. (35) such that $-(f(x) + g(x)r^{-1}(x)(g^T(x) \frac{\partial^T S_r(x)}{\partial x} - h(x)))$ is asymptotically stable about 0. Using (35), we have that

$$-\frac{\partial S_r(x)}{\partial x} \left(f(x) - g(x)r^{-1}(x)h(x) + g(x)r^{-1}(x)g^T(x) \frac{\partial^T S_r}{\partial x} + K(x)r^{-1}(x)h(x) \right) = -\frac{1}{2} h^T(x)r^{-1}(x)h(x) < 0 \quad \text{for all } x.$$

Since $S_r(x) > 0$, we conclude that $-(f(x) - g(x)r^{-1}(x)h(x) + g(x)r^{-1}(x)g^T(x) \frac{\partial^T S_r}{\partial x} + K(x)r^{-1}(x)h(x))$ is asymptotically stable about 0, making (37) asymptotically reachable from 0. Then, according to [6], for (37) there exists the controllability function \bar{L}_c defined as

$$\bar{L}_c(x) = \min_{u_1, u_2} \int_{-\infty}^0 \frac{1}{2} \left\| \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \right\|^2 dt, \quad x(0) = x_0, \quad x(-\infty) = 0,$$

$\bar{L}_c(x) > 0$, which is the unique antistabilizing solution of the Hamilton–Jacobi–Bellman equation:

$$\begin{aligned} \frac{\partial \bar{L}_c}{\partial x} (f(x) - g(x)r^{-1}(x)h(x)) + \frac{1}{2} \frac{\partial \bar{L}_c}{\partial x} g(x)r^{-1}(x)g^T(x) \frac{\partial^T \bar{L}_c}{\partial x} \\ + \frac{1}{2} \frac{\partial \bar{L}_c}{\partial x} K(x)r^{-1}(x)K^T(x) \frac{\partial^T \bar{L}_c}{\partial x} = 0, \end{aligned}$$

with $\bar{L}_c(x) = S_r(x)$. \square

In conclusion, if system (2) satisfies Assumptions 2–5, and is positive real, i.e. is passive, then the system

$$\begin{aligned}\dot{x} &= f(x) + g(x)r^{-1}(x)h(x) - g(x)r^{-1/2}(x)u_1 + K(x)r^{-1/2}(x)u_2 \\ y_1 &= -r^{-\frac{1}{2}}(x)g^T(x)\frac{\partial^T S_a(x)}{\partial x} \\ y_2 &= r^{-\frac{1}{2}}(x)h(x)\end{aligned}\quad (38)$$

is asymptotically stable, asymptotically reachable from 0 and zero-state observable with the controllability function $S_r(x) > 0$ and the observability function $S_a(x) > 0$. Hence, the required energy supply is provided by the input effort required for system (38) to reach the state x and the available stored energy is given by the output energy of system (38), respectively. Thus, the dissipation of system (2) is translated into observability/controllability of system (38). Then the positive real singular value functions of (2) are the Hankel axis singular value functions of (38), i.e., the singular value functions of the Hankel operator associated to (38), defined in Section 2.2.

3.1.2. The bounded real case

In the bounded real dissipativity case, J from Eq. (1), and $r(x)$ and $c(x)$ from Eqs. (21) and (20), respectively, are given by:

$$J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \quad r(x) = I - d^T(x)d(x), \quad c(x) = -d^T(x)h(x)$$

and the supply rate is

$$s(u, y) = \frac{1}{2}(\|u\|^2 - \|y\|^2). \quad (39)$$

Definition 6 ([26,36]). A system (2) is called bounded real if the system is dissipative with respect to the supply rate (39). \square

Linear systems case. For the linear, minimal, asymptotically stable case, the notion of bounded realness from Definition 6 is equivalently rewritten in terms of the transfer function as follows:

Definition 7 ([17,37,16]). A minimal, square, asymptotically stable system with the transfer function $G(s) = C(sI - A)^{-1}B + D$ is bounded real if it satisfies the following property:

$$I - G^T(-j\omega)G(j\omega) \geq 0, \quad \forall \omega \in \mathbb{R}.$$

If the inequalities are strict, then the system is called strictly bounded real. \square

Equivalently, (A, B, C, D) is strictly bounded real, if there exists $K > 0$ that satisfies the bounded real Riccati equation, see e.g. [16]:

$$AK + KA + (KB + C^T D)(I - D^T D)^{-1}(B^T K + D^T C) + C^T C = 0.$$

This equation admits a minimal and a maximal solution $K_{\min} > 0$ and $K_{\max} > 0$ and moreover $S_a(x) = \frac{1}{2}x^T K_{\min} x$ and $S_r(x) = \frac{1}{2}x^T K_{\max} x$.

Lemma 3 ([37]). K_{\min} and K_{\max}^{-1} are the observability and controllability Gramians of the state-space realization (A_c, B_c, C_c) , where:

$$\begin{aligned}A_c &= A - B(I - D^T D)^{-1}D^T C, \\ B_c &= [B(I - D^T D)^{-1/2}K_{\max}^{-1}C^T(I - DD^T)^{-1/2}], \\ C_c &= \begin{bmatrix} (I - DD^T)^{-1/2}C \\ (I - D^T D)^{-1/2}B^T K_{\min} \end{bmatrix}. \quad \square\end{aligned}$$

Definition 8 ([17]). A strictly bounded real system is called bounded real balanced if $K_{\min} = K_{\max}^{-1} = \text{diag}(\delta_1, \delta_2, \dots, \delta_n)$, where $1 \geq \delta_1 > \delta_2 > \dots > \delta_n > 0$ are called the bounded real singular values. \square

Let (A, B, C, D) be a strictly bounded real balanced system. Assume that $\delta_k > \delta_{k+1}$ and accordingly partition the state-space realization as:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C = [C_1 \quad C_2].$$

Theorem 12 ([14,17]). Let the reduced order model

$$G_r(s) = C_1(sI - A_{11})^{-1}B_1 + D$$

be obtained by truncation. Then (A_{11}, B_1, C_1) is asymptotically stable, minimal and strictly bounded real. \square

Nonlinear bounded real case. In the case of a bounded real system, a storage function $S(x)$ satisfies the following Hamilton–Jacobi–Bellman equation:

$$\begin{aligned}\frac{\partial S}{\partial x}f(x) + \frac{1}{2}\left(\frac{\partial S}{\partial x}g(x) + h^T(x)d(x)\right)r^{-1}(x) \\ \times \left(\frac{\partial S}{\partial x}g(x) + h^T(x)d(x)\right)^T + \frac{1}{2}h^T(x)h(x) = 0.\end{aligned}\quad (40)$$

To rewrite it in the form of (30) we need some properties from matrix theory:

Lemma 4 ([37]). If $d(x) \in \mathbb{R}^{p \times n}$ is such that $r(x)$ defined in (21) and $l(x) = I - d(x)d^T(x)$ are positive definite then the following relations hold:

$$d^T(x)r^{-1}(x) = l^{-1}(x)d^T(x)$$

and

$$d^T(x)r^{1/2}(x) = l^{1/2}(x)d^T(x). \quad \square$$

Assuming that $l(x) > 0$ and using this lemma, Eq. (40) can be equivalently rewritten as:

$$\begin{aligned}\frac{\partial S}{\partial x}(f(x) + g(x)r^{-1}(x)d^T(x)h(x)) + \frac{\partial S}{\partial x}g(x)r^{-1}(x)g^T(x)\frac{\partial^T S}{\partial x} \\ + \frac{1}{2}h^T l^{-1}(x)h(x) = 0.\end{aligned}\quad (41)$$

Remark 6. If $d(x) = 0$ then $r(x) = I > 0$ and Eq. (40) becomes as in [36]:

$$\frac{\partial S}{\partial x}f(x) + \frac{1}{2}\frac{\partial S}{\partial x}g(x)g^T(x)\frac{\partial^T S}{\partial x} + \frac{1}{2}h^T(x)h(x) = 0. \quad \square \quad (42)$$

Now, following similar lines as in Section 3.1.1 the bounded real counterparts of Theorems 10 and 11 can be obtained and are presented here without proof.

Theorem 13. Assume that system (2) is asymptotically reachable from 0, zero-state observable, bounded real as in Definition 6 and in addition Assumptions 3 and 5 are satisfied. Moreover assume that $l(x) = I - d(x)d^T(x) > 0$. Then the available storage S_a and the required supply S_r are the observability and controllability functions, respectively, of the following extended system:

$$\begin{aligned}\dot{x} &= f(x) + g(x)r^{-1}(x)d^T(x)h(x) \\ &\quad - g(x)r^{-1/2}(x)u_1 + K(x)l^{-1/2}(x)u_2 \\ y_1 &= -r^{-\frac{1}{2}}(x)g^T(x)\frac{\partial S_a(x)}{\partial x} \\ y_2 &= l^{-\frac{1}{2}}(x)h(x),\end{aligned}\quad (43)$$

where $K(x)$ satisfies $\frac{\partial S_r(x)}{\partial x}K(x) = h^T(x)$. \square

3.1.3. The general case revisited

In the general case, if the matrix $L(x)$ in Eq. (31) is positive definite, then the available storage and the required supply are the controllability and observability functions of the following extended system:

$$\begin{aligned}\dot{x} &= f(x) - g(x)r^{-1}(x)c(x) - g(x)r^{-1/2}(x)u_1 + K(x)L^{-1/2}(x)u_2 \\ y_1 &= -r^{-1/2}(x)g^T(x)\frac{\partial S_a(x)}{\partial x} \\ y_2 &= L^{-1/2}(x)\begin{bmatrix} 0 \\ h(x) \end{bmatrix},\end{aligned}\quad (44)$$

where $K(x)$ satisfies $\frac{\partial S_r(x)}{\partial x}K(x) = \begin{bmatrix} 0 \\ h(x) \end{bmatrix}$.

If $L(x)$ is not positive definite, then a realization such as (44) cannot be written. However, assuming the asymptotic stability about 0 of $\dot{x} = f(x) - g(x)r^{-1}(x)c(x)$ the available storage can be considered an observability function (different from (9)) of an extended system similar to (44). Since $L(x)$ is not positive definite, the asymptotic stability of $f(x) - g(x)r^{-1}(x)c(x)$ cannot be ensured by the storage (observability) function, since the conditions for Lyapunov stability are not satisfied. The same holds for the required supply which can be found as a controllability function of an extended system, which is assumed asymptotically stable.

Note that from a computational perspective, the equation for $K(x)$ which appears in Eq. (44) is bringing additional complexity. However, the result gives a direct relation between Hankel balancing and balancing based on the required supply and available storage, resulting in a suitable extension of the so-called positive real singular values, bounded real singular values, and their general counterpart as defined for linear systems to nonlinear systems. Hence, for balancing based on S_a and S_r , it is in general not necessary to compute the extended system. We refer to Section 4 for more details.

3.2. A factorization approach

In this section, we give a different interpretation of the available storage and the required supply, in terms of a pair of energy functions for a factorization system, which is requiring neither an asymptotic stability assumption, nor the matrix $L(x)$ to be positive definite. This is a general extension of the normalized coprime factorization idea presented in [21].

If $S(x)$ is the solution of the Hamilton–Jacobi–Bellman equation (22) then it equivalently satisfies the following equation:

$$\begin{aligned}\frac{\partial S}{\partial x} &\left(f(x) + g(x)r^{-1}(x)\left(g^T(x)\frac{\partial S}{\partial x}(x) - c(x)\right)\right) \\ &- \frac{1}{2}\frac{\partial S}{\partial x}g(x)r^{-1}(x)g^T(x)\frac{\partial S}{\partial x} + \frac{1}{2}c^T(x)r^{-1}(x)c(x) \\ &- \frac{1}{2}[0 \quad h^T(x)]J\begin{bmatrix} 0 \\ h(x) \end{bmatrix} = 0.\end{aligned}\quad (45)$$

Remark 7. In the linear case, the counterpart of (45) in different particular cases such as bounded real or LQG, can be found in, for instance, [8]. \square

We have the following result, relating the minimal and the maximal solutions of (22) or (30) and equivalently of (45) to the controllability, and observability of a closed loop system. This will lead to a factorization approach for balancing with respect to supply rate (1). First, we make the following assumption:

Assumption 6. Matrix Z defined by (29), associated to the Hamiltonian H as in (28), has no eigenvalues on the imaginary axis.

Theorem 14. Let system (2) satisfy Assumption 2 and be dissipative such that Assumptions 5 and 6 hold. Then $S_r(x) - S_a(x)$ is the controllability function of the following closed loop system:

$$\begin{aligned}\dot{x} &= f(x) + g(x)r^{-1}(x)\left(g^T(x)\frac{\partial S_a(x)}{\partial x} - c(x)\right) \\ &+ g(x)r^{-1/2}(x)v.\quad \square\end{aligned}\quad (46)$$

Proof. The assumptions combined with Theorem 6 result in the existence of $S_a(x)$ the stabilizing solution and $S_r(x)$ the antistabilizing solution of (22). Then we obtain from Eqs. (45) and (30) that

$$\begin{aligned}\frac{\partial(S_r - S_a)}{\partial x}(f(x) + g(x)u_a(x)) \\ + \frac{1}{2}\frac{\partial(S_r - S_a)}{\partial x}g(x)r^{-1}(x)g^T(x)\frac{\partial(S_r - S_a)}{\partial x} \\ = -\frac{\partial S_a}{\partial x}(f(x) + g(x)u_a(x)) + \frac{1}{2}\frac{\partial S_a}{\partial x}g(x)r^{-1/2}(x)g^T(x)\frac{\partial S_a}{\partial x} \\ - \frac{1}{2}c^T(x)r^{-1}(x)c(x) + \frac{1}{2}[0 \quad h^T(x)]J[0 \quad h^T(x)]^T = 0.\end{aligned}\quad (47)$$

Assumption 6 combined with Theorem 7 yields $S_r(x) - S_a(x) > 0$ and $S_r(0) - S_a(0) = 0$. Moreover, with

$$v = r^{-1/2}(x)g^T(x)\frac{\partial S_r(x) - S_a(x)}{\partial x}$$

we get $\dot{x} = -\left(f(x) + g(x)r^{-1}(x)\left(g^T(x)\frac{\partial S_r(x)}{\partial x} - c(x)\right)\right)$ which is asymptotically stable. Using Theorem 1, $S_r(x) - S_a(x)$ is the controllability function of the closed loop system (46). \square

Remark 8. Since the system (2) is assumed dissipative and reachable, then according to Theorem 6, system (46) is asymptotically stable, as well as the vector field $-\left(f(x) + g(x)r^{-1}(x)\left(g^T(x)\frac{\partial S_r(x)}{\partial x} - c(x)\right)\right)$. Then, according to [38, Theorem 3.1.8] the controllability function $S_a(x) - S_r(x) > 0$; a result obtained without using Assumption 6. \square

For the following line of thinking we need a slightly more general observability function definition.

Definition 9. Given a nonlinear system, we can define the following observability function, for the input equal to 0: $L_o^M(x) = \int_0^\infty \frac{1}{2}\|y\|_M^2 dt$, $x(0) = x$, $x(\infty) = 0$ for a symmetric matrix $M \in \mathbb{R}^{p \times p}$, where $\|y\|_M^2 = y^T M y$. \square

Throughout the remainder of this section L_o^M is replacing the original observability function L_o in the developments. Following the reasoning in [6], if the system is asymptotically stable and $L_o^M(x)$ exists, then L_o^M uniquely satisfies the nonlinear Lyapunov equation:

$$\frac{\partial L_o^M(x)}{\partial x}f(x) + \frac{1}{2}h^T(x)Mh(x) = 0, \quad L_o^M(0) = 0. \quad (48)$$

Next, we relate this extension of the observability function for the factorized system with the available storage function of the original system.

Theorem 15. Assume that system (2) satisfies Assumption 2, and is dissipative with respect to the supply rate (1). Then $S_a(x)$ is the

observability function L_o^M with respect to Definition 9, where $M = J$, of the following closed loop system:

$$\begin{aligned}\dot{x} &= f(x) + g(x)r^{-1}(x) \left(g^T(x) \frac{\partial^T S_a(x)}{\partial x} - c(x) \right) \\ y_1 &= r^{-1}(x) \left(g^T(x) \frac{\partial^T S_a(x)}{\partial x} - c(x) \right) \\ y_2 &= h(x) + d(x)r^{-1}(x) \left(g^T(x) \frac{\partial^T S_a(x)}{\partial x} - c(x) \right). \quad \square\end{aligned}\quad (49)$$

Proof. Since the system (2) is assumed reachable and dissipative with respect to supply rate (1), $S_a \geq 0$ exists and uniquely satisfies Eq. (45), that can be rewritten as:

$$\begin{aligned}\frac{\partial S_a(x)}{\partial x} \left(f(x) - g(x)r^{-1}(x) \left(g^T(x) \frac{\partial^T S_a(x)}{\partial x} - c(x) \right) \right) \\ + \frac{1}{2} [y_1^T \quad y_2^T] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 0,\end{aligned}$$

which is the observability Lyapunov equation (48) for system (49), with $M = J$. Since S_a is the unique stabilizing solution of (45), then system (49) is asymptotically stable and the statement of the proposition is proven. \square

Remark 9. If the dissipative system (2) is assumed zero-state observable and satisfying Assumption 3, then the observability function L_o^J of the system (49) is positive definite. \square

In conclusion, if the original dissipative system (2) satisfies Assumptions 2, 4 and 5, then S_a and S_r exist and the system:

$$\begin{aligned}\dot{x} &= f(x) + g(x)r^{-1}(x) \left(g^T(x) \frac{\partial^T S_a(x)}{\partial x} - c(x) \right) + g(x)r^{-1/2}(x)v \\ y_1 &= r^{-1}(x) \left(g^T(x) \frac{\partial^T S_a(x)}{\partial x} - c(x) \right) \\ y_2 &= h(x) + d(x)r^{-1}(x) \left(g^T(x) \frac{\partial^T S_a(x)}{\partial x} - c(x) \right)\end{aligned}\quad (50)$$

has the following set of energy functions attached: $\widehat{L}_c(x) = S_r(x) - S_a(x) > 0$ and $\widehat{L}_o^J(x) = S_a(x) > 0$, the latter with respect to Definition 9. We notice that the minimal energy required to reach a state of system (50) is related to the dissipated energy of system (2), whereas the energy observed at the output system (50) is the maximal storage available at that state in the original system (2).

4. Balancing and dissipativity preserving model reduction

In this section, we provide a relation between the singular value functions computed with respect to the balancing of S_a and S_r and the axis singular value functions of system (50), based on the results of Section 3.2. We make the following working assumption for the dissipative system (2):

Assumption 7. Assume that $S_a(x)$ and $S_r(x)$ exist such that (23) and (24) are asymptotically stable about 0 and moreover,

$$0 < S_a < S_r. \quad \square$$

Then system (50) is asymptotically stable with the energy functions $\widehat{L}_c(x) > 0$, $\widehat{L}_c(0) = 0$ and $\widehat{L}_o^J(x) > 0$, $\widehat{L}_o^J(0) = 0$. Then the line of thinking described in Section 2.2 can be followed. First we make the following working assumption upon system (50):

Assumption 8. The matrices $\frac{\partial^2 \widehat{L}_o^J}{\partial x^2}(0)$ and $\left(\frac{\partial^2 \widehat{L}_c}{\partial x^2}(0) \right)^{-1}$ are positive definite and the eigenvalues of the product of these two matrices are distinct. \square

We will split the discussion about singular value functions into two parts, according to the discussions in Sections 3.1 and 3.2.

As in Theorem 4, for the original system (2), we define the “dissipativity” axis singular values, in the coordinates $x = \xi(s)$ with respect to S_a and S_r as:

$$\pi_i^2(s) = \frac{S_a(\xi_i(s))}{S_r(\xi_i(s))}. \quad (51)$$

They express the gain between the storage supplied to reach the state $x_i = \xi_i(s)$ to 0 and the maximum storage available in the future, at the same state. This definition makes sense once we connect it to the input–output energy gain of the Hankel operator associated to the extended systems defined in Section 3.

Positive/bounded real singular value functions. Let us consider the passivity case. If $\bar{\rho}(s)$ is an axis singular value function of system (2) then, using the Eqs. (35) and (34) and the definitions of S_a and S_r , we can write for a state $\xi(s)$:

$$\begin{aligned}\bar{\rho}^2(s) &= \frac{\bar{L}_o(\xi(s))}{\bar{L}_c(\xi(s))} = \frac{\| [y_1^T(\xi(s)) \ y_2^T(\xi(s))]^T \|_{L_2}}{\| [u_1^{*T}(\xi(s)) \ u_2^{*T}(\xi(s))]^T \|_{L_2}} \\ &= \frac{S_a(\xi(s))}{S_r(\xi(s))} = \pi^2(s),\end{aligned}\quad (52)$$

$\bar{L}_c(x)$ and $\bar{L}_o(x)$ are the controllability and observability functions, respectively of (38). This means that applying the theory of Section 2.2 we call $\bar{\rho}_i(\xi(s))$ the positive real axis singular values. Based on this, the model order reduction technique of Fujimoto & Scherpen presented in Section 2.2 can be applied on system (43) or system (38).

Singular value functions, the factorization approach. If Assumption 8 is satisfied then for (50) the axis singular value functions can be defined as in Theorem 3:

$$\rho_i^2(s) = \frac{\widehat{L}_o^J(\xi_i(s))}{\widehat{L}_c(\xi_i(s))}. \quad (53)$$

According to [7], this definition is related to the Hankel norm of system (50), i.e. $\sup_s \rho(s)$ is the maximum gain between the past input energy and future output energy. The relation between the ρ ’s and the π ’s is the following:

Theorem 16. Assume that system (50) satisfies the assumptions from the preamble of Theorem 3 so that $\rho_i(s)$ exist. If $\pi_i(s)$ are the axis singular values from balancing S_a and S_r , then:

$$\pi_i(s) = \frac{\rho_i(s)}{\sqrt{1 + \rho_i^2(s)}}. \quad \square \quad (54)$$

Proof. $\rho_i^2(s) = \frac{\widehat{L}_o^J(\xi_i(s))}{\widehat{L}_c(\xi_i(s))} = \frac{S_a(\xi_i(s))}{S_r(\xi_i(s)) - S_a(\xi_i(s))} = \frac{1}{\frac{S_r(\xi_i(s))}{S_a(\xi_i(s))} - 1}$. So, we

can write: $\rho_i^2(s) = \frac{1}{\frac{S_r(\xi_i(s))}{S_a(\xi_i(s))} - 1} = \frac{\pi_i^2(s)}{1 - \pi_i^2(s)}$ and using the fact that

$\pi_i(s), \rho_i(s) > 0$ we obtain the relation (54). \square

This equation constitutes the relation between the axis singular value functions of (2) and the axis singular value functions of system (50). It simply states that the axis singular values computed with respect to the gain between the required supply and the available storage have a Hankel gain structure, making it possible to apply the balancing techniques of Section 2.2. It can be easily checked that ρ is a monotonously increasing function of π and moreover $\rho_i(s) \leq 1$ and $\pi_i(s) \leq 1$.

Remark 10. This result is in accordance with the linear coprime factorization case, the Hankel singular values of the factorization

are related in this way to the singular values of the Riccati balancing original system, see [11] for more details. Moreover, if Assumption 8 holds, then, equivalently the eigenvalues of the product $\left(\frac{\partial^2 S_r}{\partial x^2}(0)\right)^{-1} \cdot \frac{\partial^2 S_a}{\partial x^2}(0)$ are distinct and nonzero. \square

Based on Theorem 2, the Hankel singular value problem (14) of system (50) can be easily rewritten as singular value problem for the original system, in terms of the available storage and the required supply.

Proposition 2. Suppose there exists $\lambda \in \mathbb{R}$ such that the result in Theorem 2 holds for system (50). Then there exists $\nu \in \mathbb{R}$ such that

$$dS_a(x) = \nu \cdot dS_r(x), \quad (55)$$

with $\nu = \frac{\lambda}{1-\lambda}$. \square

Now we are ready to give an extension of Theorems 4 and 5 to the general dissipative case for the pair of storage functions $S_{a,r}(x)$. So, first an input-normal output-diagonal realization is presented.

Theorem 17. Assume that system (2) is dissipative and that Assumptions 7 and 8 hold. Then there exists a coordinate transformation $x = \Phi(z)$ such that:

$$S_r(\Phi(z)) = \frac{1}{2} z^T z \quad \text{and} \quad S_a(\Phi(z)) = \frac{1}{2} \sum_i z_i^2 \bar{\pi}_i^2(z_i), \quad (56)$$

where $\bar{\pi}_i(z_i) = \pi_i(\Phi_i(z))$. \square

Proof. We apply Theorem 4 to system (50). Then there exists a coordinate transformation $x = \Psi(\bar{z})$ such that $\Psi(0, \dots, s, \dots, 0) = \xi_i(s)$, where s is the i -th component (see [7]). Then $\hat{L}_0(\bar{z}) = \frac{1}{2} \sum_i \bar{z}_i \rho_i(\bar{z}_i) = S_a(\Psi(\bar{z}))$ and $\hat{L}_c(\Psi(\bar{z})) = \frac{1}{2} \sum_i \bar{z}_i^2 = S_r(\Psi(\bar{z})) - S_a(\Psi(\bar{z})) = S_r(\Psi(\bar{z})) - \frac{1}{2} \sum_i \bar{z}_i \rho_i(\bar{z}_i)$. Then

$$S_r(\Psi(\bar{z})) = \frac{1}{2} \sum_i \bar{z}_i^2 (1 + \rho_i^2(\bar{z}_i)).$$

Using relation (54) we have

$$S_r(\Psi(\bar{z}_i)) = \frac{1}{2} \bar{z}_i^2 \left(1 + \frac{\pi_i^2(\bar{z}_i)}{1 - \pi_i^2(\bar{z}_i)} \right) = \frac{1}{2} \bar{z}_i^2 \cdot \frac{1}{1 - \pi_i^2(\bar{z}_i)}.$$

Also, the available storage function can be written as: $S_a(\Psi(\bar{z}_i)) = \frac{1}{2} \bar{z}_i^2 \cdot \frac{1}{1 - \pi_i^2(\bar{z}_i)} \cdot \pi_i^2(\bar{z}_i)$. We consider the coordinate transformation $\bar{z} = \phi(z)$, defined by: $z_i = \frac{\bar{z}_i}{1 - \pi_i(\bar{z}_i)}$. Then in the $z = [z_1, z_2, \dots, z_i, \dots, z_n]^T$ coordinates we get the storage functions as in relation (56). Finally the balancing transformation is $x = \Phi(z) = \Psi(\phi(z))$. \square

Using the same line of thinking we write a balanced realization.

Theorem 18. Under the same assumptions as in the preamble of Theorem 17, there exists a coordinate transformation $x = \Phi(z)$ such that:

$$S_a(\Phi(z)) = \frac{1}{2} \sum_{i=1}^n z_i^2 \bar{\pi}_i(z_i) \quad \text{and} \quad S_r(\Phi(z)) = \frac{1}{2} \sum_{i=1}^n \frac{z_i^2}{\bar{\pi}_i(z_i)}, \quad (57)$$

where $\bar{\pi}_i(z_i) = \pi_i(\Phi_i(z))$. \square

Proof. Applying Theorem 5 on system (50) and following the line of thinking from the proof of Theorem 17 leads to the result. \square

For model reduction we assume that the system is in the form of Theorem 18. Moreover, assume there exists a k , $1 \leq k \leq n$ such

that the singular value functions $\pi(s)$ or $\rho(s)$ satisfy the following relation:

$$\max_{\pm s} \pi_k(s) > \max_{\pm s} \pi_{k+1}(s),$$

equivalent to

$$\max_{\pm s} \rho_k(s) > \max_{\pm s} \rho_{k+1}(s).$$

Then the states $z^1 = [z_1 \dots z_k]^T$ require less storage supply to be reached and they have more available storage than the states $z^2 = [z_{k+1} \dots z_n]^T$. Splitting system (2) accordingly, we get:

$$f(z) = \begin{bmatrix} f^1(z^1, z^2) \\ f^2(z^1, z^2) \end{bmatrix}, \quad g(z) = \begin{bmatrix} g^1(z^1, z^2) \\ g^2(z^1, z^2) \end{bmatrix},$$

$$h(z) = h(z_1, z_2).$$

If we truncate the states z^2 , that is we set $z^2 = 0$, we obtain two subsystems:

$$\begin{aligned} \Sigma^1: \quad & \dot{z}^1 = f^1(z^1, 0) + g^1(z^1, 0)u \\ & y^1 = h(z^1, 0) + d(z^1, 0)u, \\ \Sigma^2: \quad & \dot{z}^2 = f^2(0, z^2) + g(0, z^2)u \\ & y^2 = h(0, z^2) + d(0, z^2)u \end{aligned} \quad (58)$$

which have the following properties:

Theorem 19. The subsystems Σ^1 and Σ^2 are balanced in the sense of Theorem 18, with the following properties:

$$\begin{aligned} S_a^1(z^1) &= S_a(z^1, 0), & S_r^1(z^1) &= S_r(z^1, 0) \quad \text{and} \\ S_a^2(z^2) &= S_a(0, z^2), & S_r^2(z^2) &= S_r(0, z^2). \end{aligned} \quad (59)$$

The singular value functions of subsystem Σ^1 are $\bar{\pi}_i(z_i, 0)$, $i = 1, k$ and the singular value functions of subsystem Σ^2 are $\bar{\pi}_j(0, z_j)$, $j = k+1, n$. Moreover, $\Sigma^{1,2}$ are dissipative with respect to the supply rate $s(u, y^{1,2})$. \square

Proof. In the $z = [z^1 z^2]^T$ coordinates, $S_a(z)$ satisfies Eq. (22):

$$\begin{aligned} & \left[\frac{\partial S_a(z^1, z^2)}{\partial z^1} \frac{\partial S_a(z^1, z^2)}{\partial z^2} \right] \begin{bmatrix} f^1(z^1, z^2) \\ f^2(z^1, z^2) \end{bmatrix} \\ & + \frac{1}{2} \left(\left[\frac{\partial S_a(z^1, z^2)}{\partial z^1} \frac{\partial S_a(z^1, z^2)}{\partial z^2} \right] \begin{bmatrix} g^1(z^1, z^2) \\ g^2(z^1, z^2) \end{bmatrix} - c^T(z^1, z^2) \right) \\ & \times r^{-1}(z^1, z^2) \cdot \left(\left[\frac{\partial S_a(z^1, z^2)}{\partial z^1} \frac{\partial S_a(z^1, z^2)}{\partial z^2} \right] \begin{bmatrix} g^1(z^1, z^2) \\ g^2(z^1, z^2) \end{bmatrix} \right. \\ & \left. - c^T(z^1, z^2) \right)^T - \frac{1}{2} [0 \quad h(z^1, z^2)] \begin{bmatrix} 0 \\ h^T(z^1, z^2) \end{bmatrix} = 0. \end{aligned} \quad (60)$$

Since $S_a(z)$ is in the form (57), we have the following property from [7]:

$$z^i = 0 \Leftrightarrow \frac{\partial S_a(z)}{\partial z^i} = 0, \quad i = 1, 2 \quad (61)$$

(since $\pi_i(z_i) > 0$). Substituting $z^2 = 0$ and the property (61) in the above equation, we obtain:

$$\begin{aligned} & \frac{\partial S_a(z^1, 0)}{\partial z^1} f^1(z^1, 0) + \frac{1}{2} \left(\frac{\partial S_a(z^1, 0)}{\partial z^1} g^1(z^1, 0) - c^T(z^1, 0) \right) \\ & \times r^{-1}(z^1, 0) \left(\frac{\partial S_a(z^1, 0)}{\partial z^1} g^1(z^1, 0) - c^T(z^1, 0) \right)^T \\ & - \frac{1}{2} [0 \quad h(z^1, 0)] \begin{bmatrix} 0 \\ h^T(z^1, 0) \end{bmatrix} = 0 \end{aligned} \quad (62)$$

coinciding with the Hamilton–Jacobi–Bellman equation for the available storage $S_a^1(z^1, 0)$. The same reasoning holds for the required supply $S_r^1(z^1, 0)$. Substituting $z^1 = 0$, $S_a(0, z^2)$ and $S_r(0, z^2)$ satisfy an equation similar to (62), in the z^2 coordinates and thus relations (59) are obtained. In order to prove the dissipativity part of the theorem, we notice that in the balanced form, i.e. in the $z = [z^1 \ z^2]^T$ coordinates, the dissipative system (2) satisfies the differential dissipation inequality (4) that is written as:

$$\left[\frac{\partial S_a(z^1, z^2)}{\partial z^1} \quad \frac{\partial S_a(z^1, z^2)}{\partial z^2} \right] \cdot \left(\begin{bmatrix} f^1(z^1, z^2) \\ f^2(z^1, z^2) \end{bmatrix} + \begin{bmatrix} g^1(z^1, z^2) \\ g^2(z^1, z^2) \end{bmatrix} u \right) \leq s(u, y),$$

where $y = h(z^1, z^2) + d(z^1, z^2)u$. Applying property (61) and using relations (59), we get:

$$\frac{\partial S_a^1(z^1)}{\partial z^1} (f^1(z^1, 0) + g^1(z^1, 0)u) \leq s(u, y^1),$$

where $y^1 = h(z^1, 0) + d(z^1, 0)u$ which means that Σ^1 is dissipative with respect to the supply rate $s(u, y^1)$. The same holds for Σ^2 . \square

Remark 11. Assume that system (50) is already in balanced form in the z coordinates. The extended system corresponding to Σ^1 in (58) is given by the following equations:

$$\begin{aligned} \dot{z}^1 &= f^1(z^1, 0) + g^1(z^1, 0)r^{-1}(z^1, 0) \\ &\quad \times \left(g^{1T}(z^1, 0) \frac{\partial^T S_a(z^1, 0)}{\partial z^1} - c(z^1, 0) \right) \\ &\quad + g^1(z^1, 0)r^{-1/2}(z^1, 0)v \\ y_1 &= r^{-1}(z^1, 0) \left(g^{1T}(z^1, 0) \frac{\partial^T S_a(z^1, 0)}{\partial z^1} - c(z^1, 0) \right) \\ y_2 &= h(z^1, 0) + d(z^1, 0)r^{-1}(z^1, 0) \\ &\quad \times \left(g^{1T}(z^1, 0) \frac{\partial^T S_a(z^1, 0)}{\partial z^1} - c(z^1, 0) \right) \end{aligned} \quad (63)$$

which are obtained by truncating (50). System Σ^1 is dissipative with respect to $s(u, y^1)$ and thus the storage $S_a(z^1, 0)$ satisfies the dissipation inequality

$$\frac{\partial S_a^1(z^1)}{\partial z^1} (f^1(z^1, 0) + g^1(z^1, 0)u) \leq s(u, y^1).$$

Then applying Theorem 6, the storage function $S_a(z^1, 0)$ is the stabilizing solution of the HJB equation (62). This means that system (63) is asymptotically stable. \square

The case of HJB/copprime factorization balancing. In [21] the energy functions $K^+(x_0)$ and $K^-(x_0)$, defined by relations (7), are balanced in the case of HJB/copprime factorization systems ($s(u, y) = \|u\|^2 + \|y\|^2$). We actually balance $-S_a(x_0)$ and $S_r(x_0)$, respectively. Assuming that K^+ and K^- are in balanced form (in the z coordinates), we apply Theorem 19. We obtain a reduced order model with the past input energy function $K^+(z^1, 0) = -S_a(z^1, 0)$ and $K^-(z^1, 0) = S_r(z^1, 0)$. Noticing that $K^-(z^1, 0)$ satisfies the dissipation inequality (4) we can affirm that the reduced order obtained by HJB/copprime factorization balanced truncation is dissipative with respect to the supply rate $s(u, y^1) = \|u\|^2 + \|y^1\|^2$.

Just like at the end of Section 2.2, we mention that at the moment, there is no error bound available, except some results based on linearization. Yet, the accuracy of the reduced order model can be deduced by comparing the reduced order model and original model output responses.

From a computational point of view, we briefly discuss a possible method for the computation of a truncated reduced order

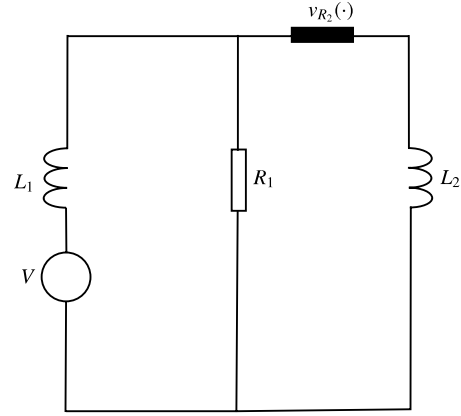


Fig. 1. RL circuit.

model. First, the storage functions, solutions of the HJB equation (22) can be approximated using, e.g. a Taylor expansion approach up to a certain order. Moreover, in some cases, the exploitation of the physical structure, or properties can provide one of the to-be-balanced storage functions. Once the storage functions are computed, one can compute the axis singular value functions based on Theorem 3, which is not a trivial problem, since a nonlinear algebraic system with a parameter is dealt with. Then using the constructive proof of Theorem 4 presented in [7] one can obtain a(n almost) balanced realization. We would like to stress the fact that the above techniques are not trivial and moreover, their application becomes more difficult with the increase in the dimension of the original model. However, as mentioned in the introduction, the example of a power system in [2] shows that a dimension of 8 may already be considered to be too large, and model order reduction should be applied.

5. Example

Next, we present an example of an RL circuit of order 2 to illustrate the results of Section 3.1, which on its turn is illustrative for the rest of the paper.

Consider an RL circuit see Fig. 1 consisting of two inductors, a linear resistor and a nonlinear, current controlled resistor. Denote by L_1 and i_1 the inductance and the current through the first inductor, by L_2 and i_2 the inductance and the current of the second inductor. R_1 , v_{R_1} and i_{R_1} are the resistance, the voltage and the current of the first resistor, respectively. Denote by $v_{R_2}(i_{R_2})$ and i_{R_2} the voltage and the current of the second (nonlinear) resistor. V is the voltage source. Using Kirchoff's voltage law one can obtain the following equations (initial conditions 0):

$$\begin{aligned} L_1 \frac{di_1}{dt} &= -v_{R_1} + V = \frac{\partial P}{\partial i_1} + V \\ L_2 \frac{di_2}{dt} &= v_{R_1} - v_{R_2}(i_{R_2}) = \frac{\partial P}{\partial i_2} \end{aligned} \quad (64)$$

with $i_{R_1} = i_1 - i_2$, and $i_{R_2} = i_2$. P is the mixed potential function, see e.g. [39] given by

$$\begin{aligned} P(i_1, i_2) &= \int_0^{i_1} v_{R_1} di_{R_1} + \int_0^{i_2} v_{R_2}(i_2) di_2 - \int_0^{i_1} R_1 i_{R_1} di_{R_1} \\ &\quad + \int_0^{i_2} v_{R_2}(i_2) di_2 = \frac{1}{2} R_1 (i_1 - i_2)^2 + \int_0^{i_2} v_{R_2}(i_2) di_2. \end{aligned}$$

Substituting this in Eqs. (64), we get:

$$\begin{aligned} \frac{di_1}{dt} &= -\frac{R_1}{L_1} (i_1 - i_2) + \frac{V}{L_1} \\ \frac{di_2}{dt} &= \frac{R_1}{L_2} (i_1 - i_2) - \frac{1}{L_2} v_{R_2}(i_2). \end{aligned} \quad (65)$$

Based on the passivity properties of the mixed potential function, [39], we propose as output

$$y = -\frac{di_1}{dt} + 3\frac{V}{L_1} = \frac{R_1}{L_1}(i_1 - i_2) + 2\frac{V}{L_1}. \quad (66)$$

Then:

$$\begin{aligned} \dot{P}(i_1, i_2) &= -R_1(i_1 - i_2)^2 - [R_1(i_1 - i_2) - v_{R_2}(i_2)]^2 \\ &\quad + \frac{V}{L_1}R_1(i_1 - i_2) \\ &= -R_1(i_1 - i_2)^2 - [R_1(i_1 - i_2) - v_{R_2}(i_2)]^2 - 2\frac{V^2}{L_1^2} + Vy \\ &\leq Vy - 2\frac{V^2}{L_1^2} \leq Vy, \end{aligned} \quad (67)$$

implying that the system is (input) strictly passive from V to y if $P \geq 0$.

Take $x_1 = i_1$, $x_2 = i_2$, $u = V$, and let $R_i = L_i = 1$, $d = 3$, $i = 1, 2$ and $v_{R_2}(i_2) = i_2 + i_2^3$, then (65) with the output (66) becomes:

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2 + u \\ \dot{x}_2 &= x_1 - x_2 - x_2^3 \\ y &= x_1 - x_2 + 2u. \end{aligned} \quad (68)$$

Denote by $x = [x_1 \ x_2]^T \in \mathbb{R}^2$ the state of system (68). $y \in \mathbb{R}$ and $u \in \mathbb{R}$ are the output and the input of (68), respectively. The mixed potential function of (68) is now given by

$$P(x) = \frac{1}{2}(x_1 - x_2)^2 + \frac{1}{4}x_2^4 \geq 0, \quad (69)$$

and thus the system is strictly passive.

Using Lukes algorithm [40], based on Taylor expansion, the following available storage function and required supply are obtained as the 4th order approximation of the solution of Eq. (34) associated for system (68) (see also e.g. [34, Chapter 6] for more details on using Lukes algorithm for approximating the solutions of a HJB equation for a passive system):

$$\begin{aligned} S_a(x) &= 0.08675x_1^2 - 0.1697x_1x_2 + 0.2094x_2^2 \\ &\quad + 0.2785963077x_1^4 - 0.9210393931x_2x_1^3 \\ &\quad + 0.905630082x_1^2x_2^2 + 0.0802107851x_1x_2^3 \\ &\quad - 0.3529193612x_2^4, \\ S_r(x) &= 0.05235x_1^2 - 0.0424x_1x_2 + 90.0217x_2^2 \\ &\quad + 0.05177923237x_1^4 + 0.1854317870x_1^3x_2 \\ &\quad - 0.2337921682x_1^2x_2^2 - 0.1021101028x_1x_2^3 \\ &\quad - 0.0061901072x_2^4. \end{aligned} \quad (70)$$

The extended system (38) is:

$$\begin{aligned} \dot{x}_1 &= -\frac{5}{4}x_1 + \frac{5}{4}x_2 + \frac{1}{2}u_1 + \frac{1}{2}K(x)u_2 \\ \dot{x}_2 &= -x_1 - x_2 - x_2^3 \\ y_1 &= -0.08675x_1 + 0.008485x_2 - 0.5571926155x_1^3 \\ &\quad + 1.38155909x_1^2x_2 - 0.950623382x_1x_2^2 \\ &\quad + 0.04010539255x_2^3 \\ y_2 &= \frac{1}{2}x_1 - \frac{1}{2}x_2, \end{aligned} \quad (71)$$

with $K(x)$ such that $\frac{\partial S_r}{\partial x} = x_1 - x_2$. System (71) is locally asymptotically stable about 0, with the controllability function $S_r(x)$ and

the observability function $S_a(x)$. Thus, the balancing procedure can be applied with respect to the S_r and S_a . The singular value functions of (68) are the Hankel singular value functions of the system (71), as described by relation (52) at the start of Section 3.2. We directly proceed with the computation of the positive real balanced realization of the original system (68). The next step after the (approximate) computation of the required supply and available storage is to perform a coordinate transformation on the system, $x = \alpha(v)$, such that, in the new coordinate we can write

$$\tilde{S}_r(v) = S_r(\alpha(v)) \approx \frac{1}{2}v^T v$$

and

$$\tilde{S}_a(v) = S_a(\alpha(v)) \approx \frac{1}{2}v^T \text{diag}(\tau_1(v), \tau_2(v))v.$$

Using again Taylor approximations, we have the expression in Box I and system (68) becomes:

$$\begin{aligned} \dot{v}_1 &= 3.84313063v_2 + 1.421722018v_1 - 11.95775153v_1^3 \\ &\quad - 154.9391819v_1^2v_2 - 339.3391697v_1v_2^2 \\ &\quad - 154.2192798v_1^2v_2 - 339.3391697v_1v_2^2 \\ &\quad - 154.2192798v_2^3 + (0.4487726741 \\ &\quad - 4.476209938v_1^2 - 13.37883062v_2^2 \\ &\quad - 17.72846781v_1v_2)u \\ \dot{v}_2 &= -1.786235805v_1 - 3.421722019v_2 - 38.23497398v_1^3 \\ &\quad - 87.9134645v_1^2v_2 + 37.9271818v_1v_2^2 + 103.206409v_2^3 \\ &\quad + (-0.2827909772 + 20.10729305v_2^2 \\ &\quad + 11.93917935v_1v_2 + 1.70778207v_1^2)u \\ y_v &= -3.16802269v_1 - 8.563646762v_2 + 28.20459034v_1^3 \\ &\quad + 108.5796921v_1^2v_2 + 123.752606v_1v_2^2 \\ &\quad + 44.065626843v_2^3 + 2u \end{aligned} \quad (72)$$

and furthermore, $\tau_1(v) \approx 12.14556887 + 653.5861792v_1^2 + 22.7936007v_1v_2 - 1926.192141v_2^2$, $\tau_2(v) \approx 1.314117050 + 21.87239925v_1^2 + 136.5697244v_1v_2 + 207.0894281v_2^2$.

Further, we compute the axis singular value functions according to Theorem 3. We approximately solve a nonlinear algebraic system with a parameter, namely s . The nonlinear algebraic system is given by the following two equations, to be solved simultaneously:

$$\tilde{S}_r(p) = \frac{1}{2}s^2 \text{ and } \det \left[\frac{\partial^T \tilde{S}_r(p)}{\partial p} \quad \frac{\partial^T \tilde{S}_a(p)}{\partial p} \right] = 0, \text{ with } p = \xi(s). \text{ We get}$$

the following coordinate transformation:

$$\begin{aligned} \xi_1 &= \begin{bmatrix} -0.4694970359s - 0.5126126872s^3 + o(s^5) \\ 0.8829340483s + 22.44651428s^3 + o(s^5) \end{bmatrix} \\ \xi_2 &= \begin{bmatrix} -0.8829340491s - 0.5126126872s^3 + o(s^5) \\ 0.469470355s - 0.9640172505s^3 + o(s^5) \end{bmatrix}. \end{aligned}$$

The axis singular value functions are computed from Theorem 3 as:

$$\begin{aligned} \rho_1(s) &= 2.506079510 + 69.19812137s^2 + o(s^4) \\ \rho_2(s) &= 0.4508902128 + 0.8176340704s^2 + o(s^4). \end{aligned}$$

The final coordinate transformation that brings the system into a positive real “input-normal/output-diagonal” form is based on the construction described in [7, Lemma 5]. This coordinate transformation, denoted by $v = \Phi(\mu)$ brings the available storage and the required supply into a(n almost) balanced form, i.e. in the new coordinates μ , we have:

$$\bar{S}_r(\mu) = \tilde{S}_r(\Phi(\mu)) \approx \frac{1}{2}(\mu_1^2 + \mu_2^2)$$

$$\alpha(v) = \begin{bmatrix} 2.228299663v_1 + 38.79691823v_1^2v_2 + 35.12598277v_1v_2^2 + 10.80219173v_1^3 \\ 5.396322362v_1 - 69.78277385v_1^2v_2 - 17.40239861v_1^3 - 88.6266232v_1v_2^2 + 8.563646762v_2 - 44.065626843v_2^3 \end{bmatrix}$$

Box I.

$$\Phi(\mu) = \begin{bmatrix} 0.46949\mu_1 - 0.8829340488\mu_2 + 0.5126127768\mu_1^3 - 0.51261277768\mu_2^3 + 4.820014038\mu_1^2\mu_2 - 6.133178718\mu_1\mu_2^2 \\ 0.8829340489\mu_1 + 0.469497035\mu_2 + 22.44651407\mu_1^3 - 0.964017287\mu_2^3 - 56.26945367\mu_1^2\mu_2 - 9.20848162\mu_1\mu_2^2 \end{bmatrix}$$

Box II.

$$\begin{aligned} \bar{S}_a(\mu) &= \tilde{S}_a(\Phi(\mu)) \approx \frac{1}{2}(\rho_1^2(\mu_1)\mu_1^2 + \rho_2^2(\mu_2)\mu_2^2) \\ &= 6.28043451\mu_1^2 + 0.203301984\mu_2^2 + 346.8319882\mu_1^4 \\ &\quad + 0.7373264\mu_2^4. \end{aligned}$$

Making use of the Taylor approximations, once more, we get the expression in Box II. Applying the coordinate transformation Φ on (72), we obtain the balanced form of system (68) with respect to the storage functions S_a and S_r :

$$\begin{aligned} \dot{\mu}_1 &= 0.2837010091\mu_2 + 5.314436042\mu_1 - 6.110937742\mu_2^3 \\ &\quad - 13.45288136\mu_1\mu_2^2 - 82.15902847\mu_1^3 \\ &\quad - 314.9116532\mu_1^2\mu_2 + (0.04603832223 \\ &\quad - 28.02554182\mu_1^2 - 2.609052266\mu_2^2 \\ &\quad + 1.409244542\mu_1\mu_2)u \\ \dot{\mu}_2 &= 0.4709920024\mu_2 + 1.773193819\mu_1 - 6.692014511\mu_2^3 \\ &\quad + 329.8316103\mu_1^3 - 65.13660482\mu_1\mu_2^2 \\ &\quad - 329.268061\mu_1^2\mu_2 + (0.2634671489 - 2.128450809\mu_2^2 \\ &\quad - 39.03304654\mu_1\mu_2 + 39.42371211\mu_1^2)u \\ y_\mu &= -9.048490286\mu_1 - 1.223451656\mu_2 - 10.68212618\mu_2^3 \\ &\quad - 393.2645935\mu_1^2\mu_2 - 109.6769272\mu_1\mu_2^2 \\ &\quad - 9417325831\mu_1^3 + 2u. \end{aligned}$$

6. Conclusions

In this paper we applied the balancing theory based on the non-linear Hankel norm approach, to the general case of dissipative systems with respect to a general quadratic supply rate. This is a unified approach that contains particular cases like positive real, bounded real, LQG, etc. The starting idea is turning the available storage and the required supply into the controllability and observability functions of new state–space realizations. The singular value functions of these systems are related to the original singular value functions defined with respect to the available storage and the required supply. Using this relation balanced realizations are provided. Truncating the more or the less dissipative states, lower order approximations are obtained. These approximations preserve the original dissipativity property. For future work we intend to solve the same problem for the case when the nonsingularity assumption (see Assumption 5 and Remark 4) does not hold.

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